# Numerical studies of a one-dimensional three-spin spin-glass model with long-range interactions

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We study a $p$-spin spin-glass model to understand if the finite-temperature glass transition found in the mean-field regime of $p$-spin models, and used to model the behavior of structural glasses, persists in the nonmean-field regime. By using a three-spin spin-glass model with long-range power-law diluted interactions we are able to continuously tune the (effective) space dimension via the exponent of the interactions. Monte Carlo simulations of the spin-glass susceptibility and the two-point finite-size correlation length show that *deep* in the nonmean-field regime, the finite-temperature transition is lost whereas this is not the case in the mean-field regime, in agreement with the prediction of Moore and Drossel [Phys. Rev. Lett. **89**, 217202 (2002)] that three-spin models are in the same universality class as an Ising spin glass in a magnetic field. However, *slightly* in the nonmean-field region, we find an apparent transition in the three-spin model, in contrast to results for the Ising spin glass in a field. This may indicate that even larger sizes are needed to probe the asymptotic behavior in this region.

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## I. INTRODUCTION

There has been considerable interest in relating structural glasses to spin glasses because spin-glass models are more amenable to analytical and numerical calculations than models of interacting atoms. This activity was started by Kirkpatrick, Thirumalai, and Wolynes[1–3] who observed a close similarity between the theory for the dynamics of $p$-spin models with $p>2$ [at the mean-field (MF) level] and mode-coupling theory[4] for the dynamics of supercooled liquids. At the mean-field level, the $p$-spin model has two transitions (for a review, see Ref. 5). There is a dynamical transition at a temperature $T=T_d$, also found in mode-coupling theory, below which ergodicity breaking occurs but which is not associated with any thermodynamic singularities. In addition, there is a transition at $T_c<T_d$ which does have thermodynamic singularities and below which replica symmetry breaking (RSB) occurs at the "one-step" level.[5] It is this transition which is associated with a possible (ideal) thermodynamic glass transition of structural glasses, where $T_c$ corresponds to the Kauzmann temperature $T_{\mathrm{K}}$.[6]

The connection between structural glasses and $p$-spin models is less clear beyond the mean-field level. The dynamical transition at $T_d$ is an artifact of the mean-field limit[7,8] since it arises from an exponentially large number of excited states which trap the system for exponentially long times, thereby preventing an infinite system reaching equilibrium. For a finite-dimensional system, however, activation over *finite* free-energy barriers restores ergodicity. Thus the only transition which *might* occur in finite-dimensional $p$-spin models and structural glasses is the thermodynamic transition at $T_c$.

Even this transition is likely to be significantly different in finite dimensions from mean-field predictions, especially for odd $p$. The reason is that odd-$p$ models violate spin-inversion symmetry ($S_i \to -S_i$ for all $i$; $S_i \in \{\pm 1\}$) so one might expect that the expectation value of the spin would be nonzero at all temperatures $T$. However, the spin average (and hence the spin-glass order parameter) is actually zero in mean-field models because of their infinite connectivity, see, for example, Ref. 9. Nevertheless, in *any* finite-dimensional models, the spin-glass order parameter would be nonzero at all $T$ and so any transition must be of the replica symmetry breaking type. In fact, one of us and Drossel[10] argue that the transition in $p$-spin models with odd $p$ is in the same universality class as an Ising ($p=2$) spin glass in a magnetic field.[11]

Because models with even $p$ have spin-inversion symmetry, which does not seem to have an analog in structural glasses, it is natural to take $p$ odd in order to represent structural glasses. In the present paper, we study numerically whether or not a thermodynamic transition occurs in a $p=3$ spin glass (and hence presumably also in a structural glass) for a range of space dimensions.

Unfortunately, it is difficult to study spin glasses numerically in high space dimensions $d$ because the number of spins $N=L^d$ increases rapidly with linear size $L$ and typically one can only study $N$ of order of a few thousand. Therefore, the range of $L$ is too limited to perform a finite-size scaling (FSS) analysis. Recently, it has been proposed[12–16] that one can avoid this difficulty by studying a model in one dimension in which the interactions depend on a power $\sigma$ of the distance.[17] Varying $\sigma$ is analogous to varying $d$ in a finite-dimensional model. In this paper, we consider values of $\sigma$ corresponding to an effective space dimension $d_{\mathrm{eff}}$ both in the mean-field ($d_{\mathrm{eff}}>6$) and nonmean-field ($d_{\mathrm{eff}}<6$) regions. Our main results are that we find a transition in the mean-field region, and no transition for $\sigma$ *well* in the nonmean-field region, consistent with our results[13,16] for the Ising spin glass in a magnetic field. However, for a value of $\sigma$ in the nonmean-field region but not far from the critical value below which mean-field behavior occurs, we find a transition, in contrast to our results for the Ising case. We shall discuss possible reasons for this discrepancy.

 

The paper is structured as follows: in Sec. II, we give some theoretical background on the connection between the transition in the $p>2$ model and that in the Ising ($p=2$) model in a magnetic field. In Sec. III, we define the one-dimensional (1D) three-spin model and describe the quantities calculated in the simulations. In Sec. IV, we briefly give some information on the numerical method and the parameters of the simulations. Our results are presented in Sec. V and our conclusions are summarized in Sec. VI.

## II. THEORETICAL BACKGROUND

The field theory associated with $p$-spin models is a cubic field theory[1–3] with the following Ginzburg-Landau-Wilson Hamiltonian:

$$\mathcal{H}_{\rm GLW} = \int d^d\mathbf{r}\left\{ \frac{t}{2}\sum_{\alpha<\beta} q_{\alpha\beta}^2(\mathbf{r}) + \frac{1}{2}\sum_{\alpha<\beta}[\boldsymbol{\nabla}_{q_{\alpha\beta}}(\mathbf{r})]^2 - \frac{w_1}{6}\mathrm{Tr}q^3(\mathbf{r}) - \frac{w_2}{3}\sum_{\alpha<\beta} q_{\alpha\beta}^3(\mathbf{r}) \right\}, \quad (1)$$

where $q_{\alpha\beta}$ is the order parameter and $\alpha$ and $\beta$ are replica indices which run from 1 to $n$, with $n\to 0$. Terms of order $q_{\alpha\beta}^4$ and higher have been omitted (and are "irrelevant" in the nonmean-field regime). At cubic order there are two terms and the ratio of their coefficients $R\equiv w_2/w_1$ plays an important role in the properties of these models at the mean-field level. When $R>1$, mean-field theory predicts[18] that there are two transitions; a dynamical transition at $T_d$ and a second transition to a state with one-step replica symmetry breaking at a lower temperature $T_c$. When $R<1$, the transitions at $T_d$ and $T_c$ no longer occur; instead there is a single transition to a state with full RSB (FRSB).

Outside the mean-field limit, one-step replica symmetry breaking, which occurs in mean field for $R>1$, is unstable against thermal fluctuations.[19] As noted in Sec. I, a FRSB transition, which occurs in mean field for $R<1$, is in the same universality class as the Ising spin glass in a magnetic field.[10] Therefore, these arguments imply that the *only* possible critical point in finite-dimensional $p$-spin models is in the same universality class as an Ising model in a magnetic field.

A $p=3$ model in which the ratio $R$ is less than unity[20] was numerically studied by Parisi, Picco, and Ritort,[7] who found evidence for a transition. When $R<1$, the effective field in the Ising spin glass in a field mapping is smaller than for $R>1$, i.e., the correlation length of the system can become very large even if there is no transition. When the correlation length becomes on the order of the system size, this finite-size effect can be mistaken for a genuine phase transition.[10] It is one of the purposes of this work to check whether this interpretation of the work of Moore and Drossel is correct, by studying a $p$-spin long-range model in one dimension where the interplay between the correlation length and the system size can be more easily investigated.

## III. MODEL AND OBSERVABLES

We consider a two-leg ladder with Ising spins $S_i$ and $T_i$ (each take values $\pm 1$) on each rung, see Fig. 1. There are $L$ rungs so $i=1,\ldots,L$. Between rungs $i$ and $j$, one can form four combinations of three spins, namely, $S_iT_iS_j$, $S_iT_iT_j$, $S_iS_jT_j$, and $T_iS_jT_j$. With a probability $p_{ij}\sim r_{ij}^{-2\sigma}$, where

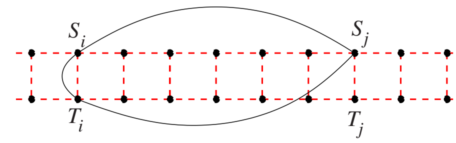


FIG. 1. (Color online) One-dimensional three-spin model. The lattice (dashed lines) consists of a two-leg ladder with an Ising spin $S_i$ at the upper end of the $i$th rung and an Ising spin $T_i$ at the lower end of the rung. An interaction couples the two spins at one rung with one of the spins at another rung. The solid line shows the interaction involving $S_i$, $T_i$, and $S_j$.

$$r_{ij} = (L/\pi)\sin(\pi|i-j|/L) \quad (2)$$

is the geometric distance between the spins arranged on a ring, each of these triplets of spins is coupled by an independent Gaussian random bond $J_{ij}^{(k)}$ with zero mean and standard deviation unity. With a $\sigma$-dependent probability $1-p_{ij}$ they are *all* zero. To avoid the probability of placing a bond being larger than 1, a short-distance cutoff is applied and thus we take

$$p_{ij} = 1 - \exp(-C/r_{ij}^{2\sigma}), \quad (3)$$

where the constant $C$ is chosen so that the mean coordination number,

$$z = \sum_{j=2}^{L} p_{1j} \quad (4)$$

takes a fixed value ($z=6$ here). The Hamiltonian is therefore given by

$$\mathcal{H} = -\sum_{i,j}\varepsilon_{ij}(J_{ij}^{(1)}S_iT_iS_j + J_{ij}^{(2)}S_iT_iT_j + J_{ij}^{(3)}S_iS_jT_j + J_{ij}^{(4)}T_iS_jT_j), \quad (5)$$

where $\varepsilon_{ij}=1$ with probability $p_{ij}$ given by Eq. (3) and zero otherwise.

We now discuss in detail the correspondence between the long-range one-dimensional model in which $\sigma$ is varied and a short-range spin-glass model in which the dimension $d$ is varied. This correspondence applies quite generally for spin-glass models. By varying $\sigma$, one can tune the model in Eq. (5) from the infinite-range to the short-range universality classes.[14,16] For $0<\sigma\le 1/2$, the model is infinite range, in the sense that $\Sigma_j[J_{ij}^2]_{\rm av}$ diverges, and for $\sigma=0$, it corresponds to the Viana-Bray model,[21] i.e., a spin glass on a random graph. For $1/2<\sigma\le 2/3$, the model describes a mean-field long-range spin glass, corresponding—within the analogy with short-range systems—to a short-range model with space dimension above the upper critical dimension $d\ge d_{\rm u}=6$. For $2/3<\sigma\le 1$, the model has nonmean-field critical behavior with a finite transition temperature $T_c$. For $\sigma\ge 1$, the transition temperature is zero. We are interested in models which are not infinite range and which have a finite $T_c$, i.e., $1/2<\sigma\le 1$.

A rough correspondence between a value of $\sigma$ in the long-range 1D Ising model and the value of a space dimension $d_{\mathrm{eff}}$ in a short-range model can be obtained by comparing the scaling of the free-energy density, $T_c\xi(T,h,d)^{-d}$, of the $d$-dimensional system to that in the 1D long-range system, $T_c\xi(T,h,\sigma)^{-1}$. When the external field $h$ is zero, $\xi\sim 1/(T-T_c)^{\nu}$, which gives a matching formula,

$$d_{\mathrm{eff}}\nu_{\mathrm{SR}}(d_{\mathrm{eff}}) = \nu_{\mathrm{LR}}(\sigma). \tag{6}$$

A second matching formula[16] is

$$d_{\mathrm{eff}} = \frac{2-\eta_{\mathrm{SR}}(d_{\mathrm{eff}})}{2\sigma-1}, \tag{7}$$

where $\eta_{\mathrm{SR}}(d_{\mathrm{eff}})$ is the critical exponent $\eta$ for the short-range model, which is zero in the MF regime. This follows from the dependence of $\xi$ on $h$ at $T_c$, $\xi\sim h^{-2/(d+2-\eta)}$, and using the fact that for the long-range system,[17]

$$2-\eta_{\mathrm{LR}} \equiv 2\sigma-1 \quad (\text{MF and nonMF regions}). \tag{8}$$

Equations (6) and (7) agree in the mean-field regime ($d>6, 1/2<\sigma<2/3$), where[17]

$$\nu_{\mathrm{LR}} = \frac{1}{2\sigma-1}, \quad \nu_{\mathrm{SR}} = \frac{1}{2} \quad (\text{MF region}) \tag{9}$$

and give

$$d_{\mathrm{eff}} = \frac{2}{2\sigma-1} \quad (\text{MF region}). \tag{10}$$

The aforementioned equations also agree to first order in $6-d$ for $d<6$ and at the lower critical dimension. Equation (7) has the following required properties: (i) $d_{\mathrm{eff}}\to\infty$ corresponds to $\sigma\to 1/2$, (ii) the upper critical dimension $d_{\mathrm{u}}=6$ corresponds to $\sigma_{\mathrm{u}}=2/3$, and (iii) the lower critical dimension, which is where $d_l-2+\eta_{\mathrm{SR}}(d_l)=0$, corresponds to $\sigma_l=1$.

To probe the existence of a transition, we compute the wave-vector-dependent spin-glass susceptibility given by

$$\chi_{\mathrm{SG}}(k) = \frac{1}{L}\sum_{i,j}[(\langle S_iS_j\rangle - \langle S_i\rangle\langle S_j\rangle)^2 + (\langle S_iT_j\rangle - \langle S_i\rangle\langle T_j\rangle)^2 + (\langle T_iS_j\rangle - \langle T_i\rangle\langle S_j\rangle)^2 + (\langle T_iT_j\rangle - \langle T_i\rangle\langle T_j\rangle)^2]_{\mathrm{av}} e^{ik(i-j)}, \tag{11}$$

where $\langle\cdots\rangle$ denotes a thermal average and $[\cdots]_{\mathrm{av}}$ an average over the disorder. To avoid bias, each thermal average is obtained from a separate copy of the spins. Therefore, we simulate four copies at each temperature. Note that the spin averages $\langle S_i\rangle$ and $\langle T_i\rangle$ are nonzero even though there is no external field because the interactions involve three spins and so the model does not have spin-inversion symmetry, as discussed in Sec. I.

The correlation length is given by[22–25]

$$\xi_L = \frac{1}{2\sin(k_{\mathrm{m}}/2)}\left[\frac{\chi_{\mathrm{SG}}(0)}{\chi_{\mathrm{SG}}(k_{\mathrm{m}})}-1\right]^{1/(2\sigma-1)}, \tag{12}$$

where $k_{\mathrm{m}}=2\pi/L$ is the smallest nonzero wave vector compatible with the boundary conditions. According to finite-size scaling,[26]

$$\frac{\xi_L}{L} = \mathcal{X}[L^{1/\nu_{\mathrm{LR}}}(T-T_c)], \quad (\sigma>2/3), \tag{13a}$$

$$\frac{\xi_L}{L^{\nu_{\mathrm{LR}}/3}} = \mathcal{X}[L^{1/3}(T-T_c)], \quad (1/2<\sigma\le 2/3), \tag{13b}$$

where $\nu_{\mathrm{LR}}$ is the correlation length exponent, given in the MF region by Eq. (9). Note, from Eq. (7) with $\eta_{\mathrm{SR}}(d_{\mathrm{eff}})=0$, which is appropriate for the MF regime, and Eq. (9), the power of $L$ in Eq. (13b) can be reexpressed in terms of $d_{\mathrm{eff}}$ according to

$$L^{1/3(2\sigma-1)} \equiv L^{d_{\mathrm{eff}}/6}, \tag{14}$$

where the factor of 6 occurs because it is the upper critical dimension $d_{\mathrm{u}}$ for spin glasses. The analogous result for ferromagnets (for which $d_{\mathrm{u}}=4$) has been verified numerically in Ref. 27. From Eq. (13), if there is a transition at $T=T_c$, data for $\xi_L/L$ ($\xi_L/L^{\nu_{\mathrm{LR}}/3}$ in the mean-field region) should cross at $T_c$ for different system sizes $L$.

We also present data for $\chi_{\mathrm{SG}}\equiv\chi_{\mathrm{SG}}(k=0)$, which has the finite-size scaling form

$$\chi_{\mathrm{SG}} = L^{2-\eta_{\mathrm{LR}}}\mathcal{C}[L^{1/\nu_{\mathrm{LR}}}(T-T_c)], \quad (\sigma>2/3), \tag{15a}$$

$$\chi_{\mathrm{SG}} = L^{1/3}\mathcal{C}[L^{1/3}(T-T_c)], \quad (1/2<\sigma\le 2/3). \tag{15b}$$

Hence, curves of $\chi_{\mathrm{SG}}/L^{2-\eta_{\mathrm{LR}}}$ ($\chi_{\mathrm{SG}}/L^{1/3}$ in the mean-field regime) should also intersect. For short-range models, Eq. (15a) is less useful than Eq. (13a) in locating $T_c$ because it involves an unknown exponent $\eta$. However, for long-range models, $\eta$ is given by Eq. (8) *exactly* even in the nonmean-field regime,[17,28,29] and so Eq. (15a) is *just as useful* as Eq. (13a) in this case.[30]

From now on, all exponents will be those of the long-range system so the subscript LR will be suppressed. If there are no corrections to scaling, the intersection temperatures for all pairs of sizes should be equal to $T_c$. However, in practice there are corrections to scaling and the intersection temperatures vary with $L$ and only tend to a constant for $L\to\infty$. Incorporating the leading correction to scaling, which is characterized by a universal correction to scaling exponent $\omega$, the intersection temperature of data for, e.g., $L$ and $2L$, $T^\star(L,2L)$, varies with $L$ as

$$T^\star(L,2L) = T_c + \frac{A}{L^{\omega+1/\nu}}, \tag{16}$$

where $A$ is a nonuniversal amplitude, see the Appendix and Refs. 31–33. Equation (16) is expected to be valid in the nonmean-field region, $2/3<\sigma<1$. Approaching the critical value of $\sigma=2/3$, one expects $\omega\to 0$. In the mean-field region, $1/2<\sigma<2/3$, the critical exponents are known but we expect corrections to Eq. (16), as discussed in the Appendix.

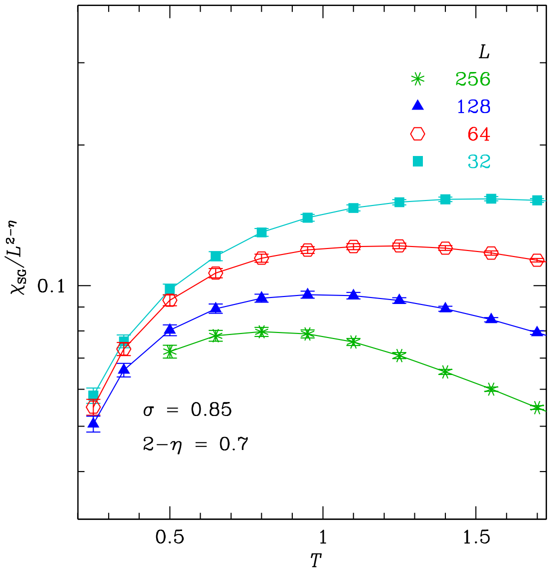


FIG. 2. (Color online) Scaled spin-glass susceptibility for $\sigma=0.85$ in which $2-\eta=2\sigma-1=0.7$. According to Eq. (15a), the data should intersect at the transition. The lack of intersections implies that there is no transition for the studied temperature range.

## IV. NUMERICAL METHOD AND EQUILIBRATION

To speed up equilibration, we use the parallel tempering (exchange) Monte Carlo method.[34] In this approach, one simulates $N_T$ copies of the spins with the same interactions, each at a different temperature between a minimum value $T_{\mathrm{min}}$ and a maximum value $T_{\mathrm{max}}$. In addition to the usual single spin flip moves for each copy, we perform global moves in which we interchange the temperatures of two copies at neighboring temperatures with a probability which satisfies the detailed balance condition. In this way, the temperature of a particular copy performs a random walk between $T_{\mathrm{min}}$ and $T_{\mathrm{max}}$, thus helping to overcome the free-energy barriers found in the simulation of glassy systems. Simulation parameters are shown in Table I.

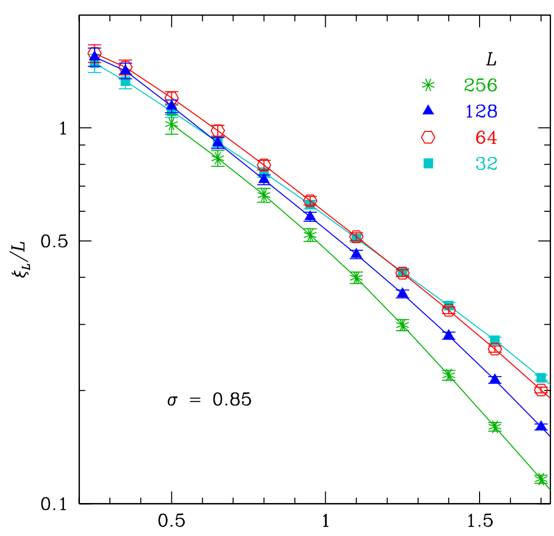


FIG. 3. (Color online) Scaled spin-glass correlation length for $\sigma=0.85$. According to Eq. (13a), the data should intersect at the transition. Although there is an intersection for the smallest pair of size, there is no intersection for the largest pair, implying the absence of a transition and in agreement with the data in Fig. 2.

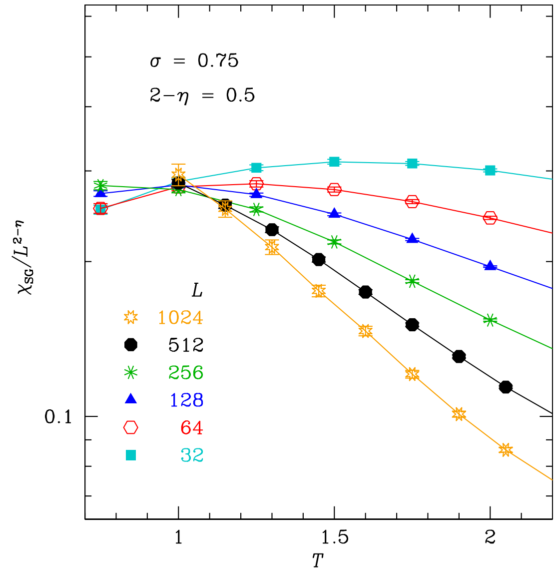


FIG. 4. (Color online) Scaled spin-glass susceptibility for $\sigma=0.75$ in which $2-\eta=2\sigma-1=0.5$.

For the simulations to be in equilibrium, the following equality must hold (see Refs. 16 and 35):

$$U=-\frac{4}{T}\left[\frac{N_b}{L}(1-\hat{q}_l)\right]_{\mathrm{av}}, \tag{17}$$

where $U$ is the energy per rung of the ladder, averaged over samples,

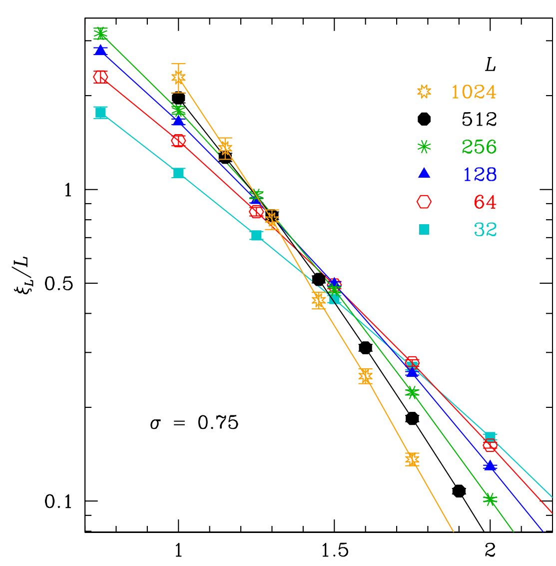


FIG. 5. (Color online) Scaled spin-glass correlation length for $\sigma=0.75$.

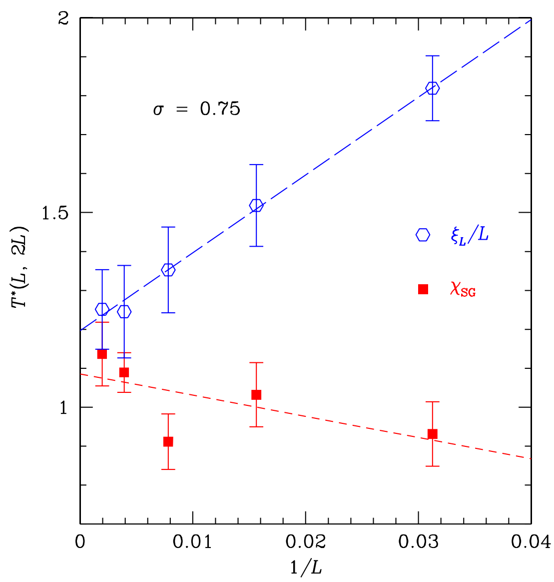


FIG. 6. (Color online) Temperatures where data sets for pairs $L$ and $2L$ intersect for $\sigma=0.75$. At large $L$, the data for both $\chi_{SG}$ and $\xi_L/L$ extrapolate to a value in the range 1.1–1.2. This implies that there is a transition at this temperature, unless the true asymptotic behavior is only seen at even larger sizes.

$$\hat{q}_l = (4N_b)^{-1}\sum_{i<j} \varepsilon_{ij}[\langle S_iS_j\rangle^2 + \langle S_iT_j\rangle^2 + \langle T_iS_j\rangle^2 + \langle T_iT_j\rangle^2] \tag{18}$$

is the link overlap of a given sample, and $N_b$ is the number of pairs of connected sites in that sample (i.e., the number of nonzero values of $\varepsilon_{ij}$). In the simulations we keep doubling the number of sweeps until Eq. (17) is satisfied within error bars. Note that Eq. (17) refers to an *average over samples*; the relationship between the energy and link overlap is *not valid* for individual samples.

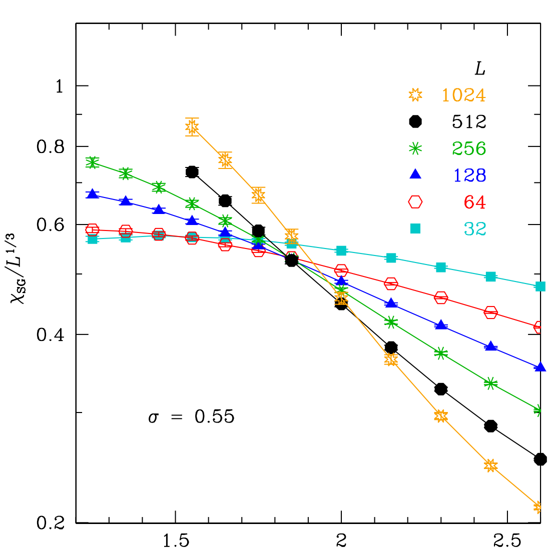


FIG. 7. (Color online) Scaled spin-glass susceptibility for $\sigma=0.55$ according to Eq. (15b). The data are consistent with a transition at $T_c \simeq 2.1$, see also Fig. 9.

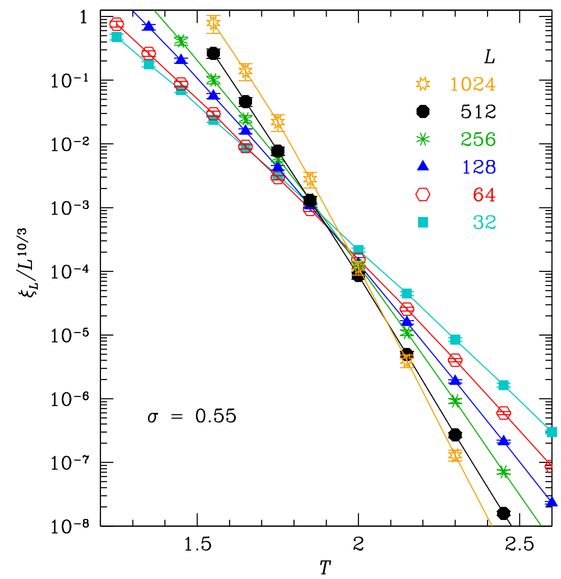


FIG. 8. (Color online) Scaled correlation length for $\sigma=0.55$. The data are consistent with a transition at $T_c \simeq 2.1$, see also Fig. 9.

## V. RESULTS

### A. $\sigma$=0.85

Results for the spin-glass susceptibility divided by $L^{2-\eta} \equiv L^{2\sigma-1} \equiv L^{0.7}$ are shown in Fig. 2 for $\sigma=0.85$ and results for the scaled correlation length are shown in Fig. 3. The $\chi_{SG}$ data show no intersections (i.e., no sign of a transition). The data for $\xi_L/L$ show an intersection for the smallest pair of

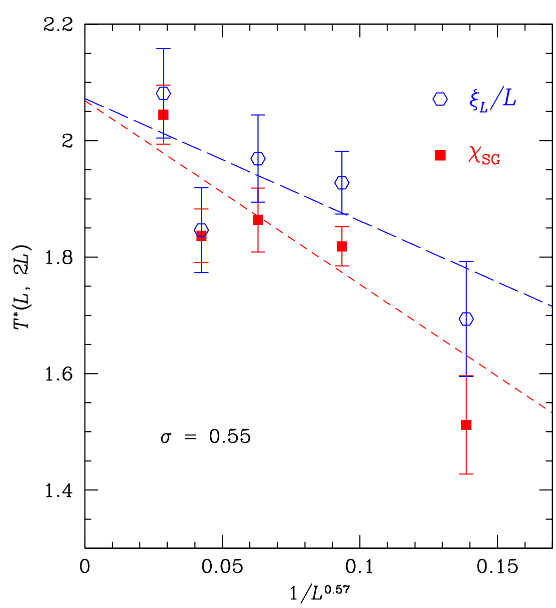


FIG. 9. (Color online) Temperatures where data for pairs $L$ and $2L$ intersect for $\sigma=0.55$. At large $L$, the data for both $\chi_{SG}$ and $\xi_L/L$ extrapolate to a value of approximately 2.1, implying that there is a transition at this temperature.

TABLE I. Parameters of the simulations for different values of $\sigma$. Here $N_{samp}$ is the number of samples, $N_{sweep}$ is the total number of Monte Carlo sweeps, $T_{min}$ and $T_{max}$ are the lowest and highest temperatures simulated, and $N_T$ is the number of temperatures. The last column shows the parameter $A$ in Eq. (3) obtained by fixing $z=6$ neighbors on average.

| $\sigma$ | $L$ | $N_{samp}$ | $N_{sweep}$ | $T_{min}$ | $T_{max}$ | $N_T$ | $A$ |
|---|---|---|---|---|---|---|---|
| 0.55 | 64 | 10000 | 65536 | 1.25 | 3.40 | 16 | 0.95527 |
| 0.55 | 128 | 10500 | 131072 | 1.25 | 3.40 | 16 | 0.81746 |
| 0.55 | 256 | 4400 | 524288 | 1.25 | 3.40 | 16 | 0.72314 |
| 0.55 | 512 | 3150 | 1048576 | 1.55 | 3.40 | 13 | 0.65411 |
| 0.55 | 1024 | 850 | 2097152 | 1.55 | 3.40 | 13 | 0.60129 |
| 0.75 | 32 | 5000 | 65536 | 0.75 | 3.25 | 11 | 2.02742 |
| 0.75 | 64 | 5000 | 131072 | 0.75 | 3.25 | 11 | 1.82345 |
| 0.75 | 128 | 16300 | 524288 | 0.75 | 3.25 | 11 | 1.71141 |
| 0.75 | 256 | 8500 | 2097152 | 0.75 | 3.25 | 11 | 1.64289 |
| 0.75 | 512 | 5600 | 16777216 | 1.00 | 3.50 | 15 | 1.59859 |
| 0.75 | 1024 | 1000 | 33554432 | 1.00 | 3.50 | 15 | 1.56903 |
| 0.85 | 32 | 5000 | 65536 | 0.25 | 4.00 | 23 | 2.65088 |
| 0.85 | 64 | 5000 | 262144 | 0.25 | 4.00 | 23 | 2.47900 |
| 0.85 | 128 | 4750 | 4194304 | 0.25 | 4.00 | 23 | 2.39485 |
| 0.85 | 256 | 3800 | 16777216 | 0.50 | 4.00 | 21 | 2.34867 |

sizes, $L=32$ and 64 but no intersection for the largest pair of sizes, $L=128$ and 256. Hence it appears that for $\sigma=0.85$, which is well in the nonmean-field regime, there is no transition. Of course, we cannot completely exclude a transition at a very low temperature.

### B. $\sigma=0.75$

Our results for $\sigma=0.75$ are shown in Figs. 4 and 5. For both $\chi_{SG}/L^{2-\eta}$ and $\xi_L/L$, we find nonzero intersection temperatures $T^{\star}(L,2L)$ which are plotted in Fig. 6. The horizontal axis in Fig. 6 is $1/L$, and, according to Eq. (16), the data would be a straight line if $1/\nu+\omega=1$. Our data are consistent with this but we do not have good enough data to obtain a precise value for this exponent. The main point is that, despite strong corrections to scaling, the data for *both* $\chi_{SG}/L^{2-\eta}$ and $\xi_L/L$ indicate a transition with $T_c$ in the range from 1.1 to 1.2.

This is rather surprising since it has been argued[10] that the transition is in the same universality class as the Ising spin glass in a magnetic field, and *no* transition has been found for that model with $\sigma=0.75$ in work by some of us.[13,16] However, corrections to scaling are very large (see Figs. 2–9), and so it is plausible that system sizes considerably larger than $L=1024$ are needed to see the true thermodynamic behavior of the three-spin model when $\sigma\searrow 2/3$, in which case there would be no inconsistency with the work of Refs. 13 and 16.

### C. $\sigma=0.55$

Our results for $\sigma=0.55$ (mean-field regime) are shown in Figs. 7 and 8. As discussed in the Appendix, the intersection temperatures in the mean-field regime are given by Eq. (A3). For $\sigma=0.55$, the exponent $5/3-2\sigma$ is equal to 0.57. We therefore plot the intersection temperatures against $1/L^{0.57}$ in Fig. 9. The data strongly suggest that there is a transition at $T_c\simeq 2.1$. This result is consistent with our earlier results for the Ising spin glass in a magnetic field,[13,16] where we also found a transition in the mean-field region.

## VI. SUMMARY AND CONCLUSION

We have studied the existence of phase transitions in a three-spin spin-glass model, that is argued to be an appropriate model to describe the (possible) ideal glass transition in a supercooled liquid. We have studied three values of the parameter $\sigma$: (i) $\sigma=0.55$ (mean-field regime), (ii) $\sigma=0.75$ (nonmean-field region but close to the mean-field boundary at 2/3), and (iii) $\sigma=0.85$ (deep inside the nonmean-field regime). Moore and Drossel[10] argue that any transition in this model is in the same universality class as that of the Ising spin glass in a magnetic field. In particular, the two models should have the same critical value of sigma where the transition disappears (corresponding to the lower critical dimension for the short-range case). In other words, if one model has a transition the other should have one and vice versa.

For the mean-field case, $\sigma=0.55$, we find a finite-temperature transition. Comparing with our previous work[13,16] for the Ising spin glass in a magnetic field, in which we also find a transition in the mean-field regime, this result is seen to be consistent with the predictions of Ref. 10.

For the case studied that is *well* in the nonmean-field regime, $\sigma=0.85$, we find no transition, in agreement with our work for the Ising spin glass in a magnetic field. This implies that there is no ideal glass transition in three dimensions since $d=3$ is *well* below the upper critical dimension of $d=6$ for models with cubic interactions, such as that in Eq. (1).

However, for $\sigma=0.75$ the results presented here, which indicate a finite transition temperature, appear to be at odds with our results for the Ising transition in a field,[13,16] where we find no transition. We note, however, that Leuzzi *et al.*[15] argue that there *is* a transition for this case, based on a nonstandard finite-size scaling analysis. In the absence of a transition, the system breaks up into domains of size $\ell$ (Imry-Ma length) which can be large at low temperatures, depending on the model. A possible explanation of our results for $\sigma=0.75$ is that $\ell(T\to 0)$ is greater than the largest system size, namely, $L=1024$, for the three-spin model, although not for the Ising model in a field studied in Ref. 16. If this is the case, even larger values of $L$ are needed to determine the asymptotic behavior of the three-spin model.

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## APPENDIX: SIZE DEPENDENCE OF INTERSECTION TEMPERATURES

According to standard finite-size scaling, the spin-glass susceptibility varies near the critical point according to

$$\chi_{\rm SG}(t,L) = L^{a}[f(L^{b}t) + L^{-\omega}g(L^{y}t) + \cdots] + c_0 + c_1 t + \cdots, \tag{A1}$$

where $t=T-T_c$. The $L^{-\omega}$ term is the leading *singular* correction to scaling and $c_0$ is the leading *analytic* correction to scaling.

*Nonmean-field regime*. In the nonmean-field regime, $\sigma_c < \sigma < 1$ with $\sigma_c=2/3$, we have $a=2-\eta=2\sigma-1$ and $b=1/\nu$. (In this section all exponents refer to the long-range universality class.) We use Eq. (A1) to calculate the temperature $T^{\star}(L,2L)$, where data for $\chi_{\rm SG}/L^{a}$ for sizes $L$ and $2L$ intersect. Expanding $f(x)$ to first order in $x$, replacing $g(x)$ by $g(0)$, and assuming that $a>\omega$ (which is certainly true near $\sigma=2/3$, where $\omega\to 0$) we recover Eq. (16).

*Mean-field regime*. Curiously, the situation in the mean-field regime, $1/2<\sigma\le 2/3$, is more complicated. First of all, the exponents $a$ and $b$ are independent of $\sigma$ (Refs. 27, 36, and 37) and take the value at $\sigma_c$ for all $1/2<\sigma<\sigma_c$, i.e., $a=b=1/3$. Second, although the $L^{2\sigma-1}$ term is replaced as the *largest* term by an $L^{1/3}$ term (due to the presence of a "dangerous irrelevant variable," cf. Refs. 27, 36, and 37) we expect this term to not disappear but rather become a *correction to scaling*. Hence, we replace Eq. (A1) by

$$\chi_{\rm SG}(t,L) = L^{1/3}[f(L^{1/3}t) + L^{-\omega}g(L^{1/3}t) + \cdots] + d_0 L^{2\sigma-1} hg(L^{1/3}t) + c_0 + \cdots \quad (1/2<\sigma<2/3). \tag{A2}$$

The correction exponent $\omega$ can be obtained in the mean-field regime from the work of Kotliar *et al.*[17] and is[38] given by $\omega=2-3\sigma$.

For $\sigma<\sigma_c$, we find that the $L^{2\sigma-1}$ term gives the leading correction in Eq. (A2) and, as a result, Eq. (16) is replaced by

$$T^{\star}(L,2L) = T_c + \frac{A'}{L^{5/3-2\sigma}}, \quad (1/2<\sigma<2/3). \tag{A3}$$

To determine the intersection temperatures of the correlation length we also need the FSS scaling form for $\chi_{\rm SG}(k_{\rm m})$. We find that there is an additional correction which dominates for $\sigma<7/12=0.5833$. However, for the value $\sigma=0.55$ used in the simulations, the resulting difference from Eq. (A3) is very small and therefore we neglect it.


1. T. R. Kirkpatrick and P. G. Wolynes, Phys. Rev. A **35**, 3072 (1987).
2. T. R. Kirkpatrick and D. Thirumalai, Phys. Rev. B **36**, 5388 (1987).
3. T. R. Kirkpatrick and P. G. Wolynes, Phys. Rev. B **36**, 8552 (1987).
4. W. Gotze and L. Sjogren, Rep. Prog. Phys. **55**, 241 (1992).
5. J.-P. Bouchaud, L. F. Cugliandolo, J. Kurchan, and M. Mézard, in *Spin Glasses and Random Fields*, edited by A. P. Young (World Scientific, Singapore, 1998).
6. W. Kauzmann, Chem. Rev. (Washington, D.C.) **43**, 219 (1948).
7. G. Parisi, M. Picco, and F. Ritort, Phys. Rev. E **60**, 58 (1999).
8. H. Bokil, B. Drossel, and M. A. Moore, Phys. Rev. B **62**, 946 (2000).
9. T. Castellani, F. Krzakala, and F. Ricci-Tersenghi, Eur. Phys. J. B **47**, 99 (2005).
10. M. A. Moore and B. Drossel, Phys. Rev. Lett. **89**, 217202 (2002).
11. J. R. L. de Almeida and D. J. Thouless, J. Phys. A **11**, 983 (1978).
12. H. G. Katzgraber and A. P. Young, Phys. Rev. B **67**, 134410 (2003).
13. H. G. Katzgraber and A. P. Young, Phys. Rev. B **72**, 184416 (2005).
14. L. Leuzzi, G. Parisi, F. Ricci-Tersenghi, and J. J. Ruiz-Lorenzo, Phys. Rev. Lett. **101**, 107203 (2008).
15. L. Leuzzi, G. Parisi, F. Ricci-Tersenghi, and J. J. Ruiz-Lorenzo, Phys. Rev. Lett. **103**, 267201 (2009).
16. H. G. Katzgraber, D. Larson, and A. P. Young, Phys. Rev. Lett. **102**, 177205 (2009).
17. G. Kotliar, P. W. Anderson, and D. L. Stein, Phys. Rev. B **27**, 602 (1983).
18. D. J. Gross, I. Kanter, and H. Sompolinsky, Phys. Rev. Lett. **55**, 304 (1985).

[19] M. A. Moore, Phys. Rev. Lett. **96**, 137202 (2006).
[20] The model studied in Ref. 7 has $R=0.88$; J. Yeo (private communication).
[21] L. Viana and A. J. Bray, J. Phys. C **18**, 3037 (1985).
[22] F. Cooper, B. Freedman, and D. Preston, Nucl. Phys. B **210**, 210 (1982).
[23] M. Palassini and S. Caracciolo, Phys. Rev. Lett. **82**, 5128 (1999).
[24] H. G. Ballesteros, A. Cruz, L. A. Fernandez, V. Martin-Mayor, J. Pech, J. J. Ruiz-Lorenzo, A. Tarancon, P. Tellez, C. L. Ullod, and C. Ungil, Phys. Rev. B **62**, 14237 (2000).
[25] D. Amit and V. Matin-Mayor, *Field Theory, The Renormalization Group and Critical Phenomena* (World Scientific, Singapore, 2005).
[26] For a discussion of how finite-size scaling is modified in the region of mean-field exponents, see, for example, Refs. 27, 36, 37, 39, and 40.
[27] J. L. Jones and A. P. Young, Phys. Rev. B **71**, 174438 (2005).
[28] M. E. Fisher, S.-k. Ma, and B. G. Nickel, Phys. Rev. Lett. **29**, 917 (1972).
[29] D. S. Fisher and D. A. Huse, Phys. Rev. B **38**, 386 (1988).
[30] One can alternatively reexpress the FSS results in Eqs. (6), (7), and (15) in the following way. A *thermodynamic* quantity $X$ has the finite-size scaling form $X=L^{y_X}\tilde{X}[L^{y_T}(T-t_c),L^{y_H}h]$, where $y_{\mathrm{T}}=1/\nu$ and $y_{\mathrm{H}}=(d+2-\eta)/2$ are the thermal and magnetic exponents. The connection between LR and SR exponents is obtained by writing this in terms of the total number of spins $N$ and equating the exponents, i.e., $y_{\mathrm{T,LR}}=y_{\mathrm{T,SR}}/d_{\mathrm{eff}}, y_{H,\mathrm{LR}}=y_{H,\mathrm{SR}}/d_{\mathrm{eff}}, y_{X,\mathrm{LR}}=y_{X,\mathrm{SR}}/d_{\mathrm{eff}}$. In the same way, the analogous result for the correction to scaling exponent $\omega$ is $\omega_{\mathrm{LR}}=\omega_{\mathrm{SR}}/d_{\mathrm{eff}}$. The relation involving $y_X$ does not apply for the correlation length since it refers to a linear dimension, rather than a volume, and so there is no factor of $d_{\mathrm{eff}}$, see Eq. (13).
[31] K. Binder, Z. Phys. B: Condens. Matter **43**, 119 (1981).
[32] H. G. Ballesteros, L. A. Fernández, V. Martín-Mayor, and A. Muñoz Sudupe, Phys. Lett. B **387**, 125 (1996).
[33] M. Hasenbusch, A. Pelissetto, and E. Vicari, Phys. Rev. B **78**, 214205 (2008).
[34] K. Hukushima and K. Nemoto, J. Phys. Soc. Jpn. **65**, 1604 (1996).
[35] H. G. Katzgraber, M. Palassini, and A. P. Young, Phys. Rev. B **63**, 184422 (2001).
[36] K. Binder, M. Nauenberg, V. Privman, and A. P. Young, Phys. Rev. B **31**, 1498 (1985).
[37] E. Luijten, K. Binder, and H. W. J. Blöte, Eur. Phys. J. B **9**, 289 (1999).
[38] This expression vanishes for $\sigma=\sigma_c=2/3$ as expected. Furthermore, using the connection between $\sigma$ and the effective dimension of a short-range model in Eq. (10), we have $\omega=\omega_{\mathrm{SR}}/d$ with $\omega_{\mathrm{SR}}=(d-6)/2$, as expected at the trivial Gaussian fixed point in a cubic field theory (Ref. 41).
[39] E. Brézin, J. Phys. France **43**, 15-22 (1982).
[40] E. Brézin and J. Zinn-Justin, Nucl. Phys. B **257**, 867 (1985).
[41] A. B. Harris, T. C. Lubensky, and J.-H. Chen, Phys. Rev. Lett. **36**, 415 (1976).